Completeness

Definition

Let $X_1, X_{2,} X_3....X_n$ be a random sample from the density f(x;θ) and $T(\underline{X})$ be a statistic . then family of densities of "T" is define to be complete iff for an arbitrary function "h" such that

$E(h(t)) = 0$

$P(h(t)) = 1$

Where h(t) is a statistic.

Or

Let $X_1, X_{2,} X_3....X_n$ be a random sample from the density f(x;θ) with parameter space $\underline{\bar{\theta}}$ and let T=t($X_1, X_{2,} X_3....X_n$) be a statistic . the family of densities of "T" is define to be complete iff

$E(h(t)) = 0$ for all θ ε $\underline{\bar{\theta}}$

Implies that

$P(h(t)) \cong 1$

Where h(t) is a statistic.

The statistic" is said to be complete iff its family of densities is complete

Procedure

First of all we write the p.d.f and convert it into the form of statistic function .

Such that function involve the statistic which is proved to be complete statistic.

Then by definition

$E(h(t)) = 0$

$\sum h(t) f(t) = 0$

Where f(t) is a function of statistic .

If $\sum (h(t)) = 0$ which is possible only when h(t)=0 the statistic "t" is said to be complete.

In case of continuous distribution

$E(h(t)) = 0$

$\int h(t) f(t) dt$

Or

$\int h(t) f(T = t) dy_t$

Then

$\int h(t) dy_t = 0$

After taking the derivative and obtain h(t)=0 .

Joint completeness

Let $X_1, X_{2,} X_3 ....X_n$ be a random sample from the density f(x; $\theta_1, \theta_2 ,...\theta_n$) and T($X_1, X_{2,} X_3 ....X_k$)

be a set of statistics.

$T_1, T_{2,} T_3 ....T_m (T_K)$ are defined to be joint complete iff

$E[h(t_1, t_{2,} t_3 ....t_k (t_m)] = 0$ for all θ ε $\underline{\overline{\theta}}$

Implies that

$E[h(t_1, t_{2,} t_3 ....t_k (t_m) = \theta_1, \theta_2 ,...\theta_n] = 1$

Where $h(t_1, t_{2,} t_3 ....t_n)$ is a statistic.

Complete minimal sufficient statistic

Let $X_1, X_{2,} X_3 ....X_n$ be a random sample from the density f(x;θ). If f(x;θ)=a(θ) b(θ) Exp[c(θ) dx] that is f(X;θ) is a number of one parameter exponential family . Then $\sum d(x)$ is a complete minimal sufficient statistic

Q.No.1

Let $X_1, X_{2,} X_3 ....X_n$ be a random sample from the density of pission distribution show that the statistic T=$\sum X_i$ is complete.

Solution

As X ~ f(X;θ)

$$f(x) = \frac{e^{-\theta}\theta^x}{x!} \qquad X: 0,1,2,. . . .,\infty$$

Then

$$X_1 + X_2 + ... + X_n = \sum x = T \sim p(n\theta)$$

$$f(t) = \frac{e^{-n\theta}(n\theta)^t}{t!} \qquad t: 0,1,2,. . . .,\infty$$

By definition of completeness

$$E(h(t)) = 0$$

$$\sum h(t) f(t) = 0$$

$$\sum h(t) \frac{e^{-n\theta}(n\theta)^t}{t!} = 0$$

$$\sum h(t) \frac{e^{-n\theta}(n)^t(\theta)^t}{t!} = 0$$

Hence $e^{-n\theta} \neq 0, (n)^t \neq 0, (\theta)^t \neq 0, t! \neq 0$

$$only \sum h(t) = 0$$

Hence it is proved T=$\sum X$ is complete in possion distribution.

Q.No.2

Let $X_1, X_{2,} X_3 ....X_n$ be a random sample from the Bernoulli distribution then show that T= $\sum X_i$ is complete.

Solution

As X~ Bernoulli (θ)

$f(x) = (c_x^1)\theta^x (1-\theta)^{1-x}$ x:0,1

Then

$X_1 + X_2 + ...X_n = \sum x = T \sim b(n, \theta)$

$f(t) = (c_t^n)\theta^t (1-\theta)^{n-t}$ t:0,1,2. . . .n

By definition of completeness

$E(h(t)) = 0$

$\sum h(t) f(t) = 0$

$\sum h(t)(c_t^n)\theta^t (1-\theta)^{n-t} = 0$

$then (c_t^n) \neq, \theta^t \neq 0, (1-\theta)^{n-t} \neq 0$

$only \sum h(t) = 0$

Hence it is proved T= $\sum X_i$ is complete for Bernulli distribution.

Q.No.3

Let $Y_{(1)} < Y_{(2)} < Y_{(3)} < .....Y_{(n)}$ be a random sample from uniform distribution with parameter "θ".show that $Y_{(n)}$ is complete.

Solution

As $Y_i \sim$ uniform (θ)

$f(y_i) = \frac{1}{\theta}$ $0<Y_i<\theta$

As we know that the p.d.f of rth order statistic

$$g(y_i) = \frac{n!}{(i-1)!(n-i)!} [F(y_{(i)})]^{i-1} [1-F(y_{(i)})]^{n-i} f(y_{(i)})$$

Put i=n , n=n

$$g(y_{(n)}) = \frac{n!}{(n-1)!(n-n)!} [F(y_{(n)})]^{n-1} [1-F(y_{(n)})]^{n-n} f(y_{(n)})$$

$$g(y_{(n)}) = \frac{n!}{(n-1)!(n-n)!} [F(y_{(n)})]^{n-1} f(y_{(n)})$$

$$g(y_{(n)}) = n[F(y_{(n)})]^{n-1} f(y_{(n)}) \qquad \text{(A)}$$

As

$$f(y_i) = \frac{1}{\theta}$$

$$F(y_i) = \int_0^{y_i} \frac{1}{\theta} dy_i$$

$$= \frac{1}{\theta} \int_0^{y_i} 1 dy$$

$$= \frac{Y}{\theta} \Big|_0^{y_i}$$

$$= \frac{y_i}{\theta}$$ s

$$F(y_{(n)}) = \frac{y_{(n)}}{\theta}$$

$$f(y_{(n)}) = \frac{1}{\theta}$$

Then (A) become

$$g(y_{(n)}) = n(\frac{y_{(n)}}{\theta})^{n-1} \frac{1}{\theta}$$

$$g(y_{(n)}) = \frac{n}{\theta} \frac{(y_{(n)})^{n-1}}{(\theta)^{n-1}}$$

$$g(y_{(n)}) = n\theta^{-n} (y_{(n)})^{n-1}$$

By definition of completeness

$$E(h(t)) = 0$$

$$\int h(t) g(y_{(n)}) dy_{(n)} = 0$$

$$\int h(y_{(n)}) n\theta^{-n} (y_{(n)})^{n-1} dy_{(n)} = 0$$

Hence $n \neq 0, (\theta)^{-n} \neq 0, (y_{(n)})^{n-1} \neq 0$

$$then \int h(y_{(n)}) dy_{(n)} = 0$$

Differentiate w.r.t $y_{(n)}$ we get

h $y_{(n)}$=0

Hence it is proved that $y_{(n)}$ is complete.

Importance of Rao Blackwell Theorem/Use /Application/ Short note on Rao Blackwell

This theorem help us a great deal in our search for minimum variance unbiased estimator, since it essentially tell us that we need to look only an unbiased estimator that are a function of sufficient estimators that are a function of sufficient statistics. However, this theorem dost not give the final answer, since there may possible be many estimator each based on sufficient statistic and each unbiased. This theorem does not throw any light to us , which of these has got the minimum variance. If the density of the sufficient statistic is complete, then there is only one unbiased estimator which is based on the sufficient statistic,then it must be the minimum variance unbiased estimator(MVUE).

Rao Blackwell theorem

Statement:

Let $X_1, X_2, \ldots\ldots, X_n$ be a random sample from the density $f(X;\theta)$.Let $T(\underline{X})$ be an unbiased estimator of θ. Let $\tau(\theta)$ & $S(\underline{X})$ is a sufficient statistic for parameter θ.

$$T'(S) = E[{}^{T(\underline{X})}\!/\!_{S(\underline{X})}]$$

Then

i) $E[T'(S)] = \tau(\theta)$

ii) $\mathrm{var}[T'(S)] \le \mathrm{var}[T(\underline{x})]$

Proof:

As $S(\underline{X})$ is sufficient then $T'(S) = E[{}^{T}\!/\!_{S}]$ .It is independent form parameter θ . Since ‘T’ is a function of “S”.

Now applying expectation

$$E[T'(S)] = E[{}^{T}\!/\!_{S}]$$

$$E[T'(S)] = E[T]$$

$$E[T'(S)] = \tau(\theta)$$

Hence first part of theorem is proved.

ii) Now variance,

Let by definition of variance

$$\mathrm{var}[T] = E[T - E(T)]^2$$

$$\mathrm{var}[T] = E[T - \tau(\theta)]^2 \qquad \therefore E(T) = T(\theta)$$

Adding & subtracting T'

$$\mathrm{var}[T] = E[T - T' + T' - \tau(\theta)]^2$$

$$\mathrm{var}[T] = E[(T - T') + (T' - \tau(\theta))]^2$$

$$\mathrm{var}[T] = E[(T - T')^2 + (T' - \tau(\theta))^2 + 2(T - T')(T' - \tau(\theta))]$$

$\text{var}[T] = E(T - T')^2 + E(T' - \tau(\theta))^2 + 2E(T - T')(T' - \tau(\theta))$

If T and T' are independent so cross product term will be vanish

$\text{var}[T] = E(T - T')^2 + E(T' - \tau(\theta))^2 + 2(0)$

$\text{var}[T] = V(T') + E(T' - \tau(\theta))^2$

$\text{var}(T) = \text{var}(T')$ + Some positive quantity

Then

$\text{var}(T) \geq \text{var}(T')$

$\text{var}(\underline{X}) \geq \text{var}(T'(S))$

Hence the required result.

Statement:

Let $X_1, X_2, ..., X_n$ be a random sample from the density $f(X;\theta)$. If $S(\underline{X})$ is complete statistic for 'θ' and $T(\underline{X})$ is an unbiased estimator of $\tau(\theta)$. Then $T'(S) = E[T(\underline{X})/S(\underline{X})]$ is "UMVUE" FOR $\tau(\theta)$.

PROOF:

Suppose $T''(S)$ be any other unbiased estimstor $\tau(\theta)$ of which is the function of $S(\underline{X})$.

Then

$E[T' - T''] = E(T') - E(T'')$

$E[T' - T''] = \tau(\theta) - \tau(\theta)$

$= 0$

And $T' = T''$ is also function of "S"

By completeness

$p[T' - T'' = 0] = 1$

$p[T' - T''] = 1$

We can say $T' - T''$. Hence there is only one unbiased estimator of $\tau(\theta)$ which is the function of "S"

Hence T' or T'' which is equal to $E[T/S]$ is an unbiased estimator of $\tau(\theta)$ that is the function of "S".

By Rao Blackwell theorem T' is an unbiased estimator of $\tau(\theta)$ and the variance is

$\text{var}(T) \geq \text{var}(T')$

So $T'(S)$ is UMVUE for $\tau(\theta)$

Cramer-raw-inequality

Question: state and prove Cramer raw inequality

Statement:

Let $t(x_1, x_2, x_3, ...x_n)$ is an unbiased estimator of $\tau(\theta)$ and $\text{var}(T) \geq {}^{[\tau'(\theta)]^2}\!/\!_{n(\theta)}]$

Where $n_i(\theta) = E[\frac{\partial \log f(\underline{X}:\theta)}{\partial\theta}]^2$ Or $\text{var}(T) \geq {}^{[\tau'(\theta)]^2}\!/\!_{E[\frac{\partial \log f(\underline{X}:\theta)}{\partial\theta}]^2}$

Then this inequality is known as Cramer raw inequality and $\text{var}(T)$ is called minimum variance bound.

Or

Question

If $\hat{\theta}$ is an unbiased estimator of $\tau(\theta)$ then prove that

$$\text{var}(\hat{\theta}) \geq \frac{[\tau'(\theta)]^2}{E\left[\frac{\partial \log f(\underset{-}{x};\theta)}{\partial\theta}\right]^2}$$

Solution

Let we have a random sample of size 'n' from a density f(x, θ).Then likelihood function will be

$$L(\underline{X}) = f(\underset{-}{x};\theta) = \prod_{i=1}^{n}(\underset{-}{x},\theta)$$

We know that Area under the curve is unity

$$\int L(\underline{X})dx = 1 \qquad \textbf{(A)}$$

It is given that $\hat{\theta}$=T is an unbiased estimator of $\tau(\theta)$

i. e E(T) = $\tau(\theta)$

E$\left[t(x_1,x_2,x_3,...,x_n)\right] = \tau(\theta)$

$$\int t(x_1,x_2,x_3,...,x_n) f((x_1,x_2,x_3,...,x_n;\theta)dx = \tau(\theta)$$

$$\int T(\underline{X})) f(\underline{x};\theta)dx = \tau(\theta)$$

$$\int T(\underset{-}{x})L(\underset{-}{x})dx = \tau(\theta) \qquad \text{(B)}$$

Now partially differentiate w.r.t 'θ' to eq (A)

$$\frac{\partial}{\partial\theta}\int L(\underline{X})dx = \frac{\partial}{\partial\theta}(1)$$

$$\int \frac{\partial}{\partial\theta} L(\underline{X})dx = 0$$

Multiply and divide by $L(\underline{X};\theta)$

$$\int \frac{1}{L(\underline{X};\theta)} \frac{\partial}{\partial\theta} L(\underline{X};\theta) L(\underline{X};\theta) dx = 0$$

$$\int \left\{ \frac{\partial}{\partial\theta} . \log L(\underline{X};\theta) \right\} L(\underline{X};\theta) dx = 0$$

$$E\left[\frac{\partial}{\partial\theta} L(\underline{X};\theta) \right] = 0$$

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Now again partially differentiate w.r.t 'θ' to eq (B)

$$\frac{\partial}{\partial\theta} \int T(\underline{X})) L(\underline{X};\theta) dx = \frac{\partial}{\partial\theta} \tau(\theta)$$

$$\int T(\underline{X}) \frac{\partial}{\partial\theta} L(\underline{X};\theta) dx = \tau(\theta)$$

Multiply and divide by $L(\underline{x};\theta)$

$$\int T(\underline{X}) \frac{1}{L(\underline{X};\theta)} \frac{\partial}{\partial\theta} L(\underline{X};\theta)..L(\underline{X};\theta) dx = \tau'(\theta)$$

$$\int T(\underline{X}) \frac{\partial}{\partial\theta} \log L(\underline{X};\theta)..L(\underline{X};\theta) dx = \tau'(\theta)$$

$$E[T(\underline{X}) \frac{\partial}{\partial\theta} \log L(\underline{X};\theta)] = \tau'(\theta)$$

$$E[YZ] = \tau'(\theta) \qquad \text{(D)}$$

Where,

$$Y = T(\underline{X})$$

$$Z = \frac{\partial}{\partial\theta} \log L(\underline{X};\theta)$$

And

Var (z) = $E(z)^2 - (E(z))^2$

From eq (C) E (z) =0

$$\text{var}(z) = E\left[\frac{\partial}{\partial\theta} \log L(\underline{X};\theta) \right]^2 - (0)^2$$

$$= E\left[\frac{\partial}{\partial\theta} \log L(\underline{X};\theta) \right]^2$$

Now

$$\text{cov}(yz) = E(yz) - E(y)E(z)$$

$$= \tau'(\theta)$$

From eq (C)

E(Z)=0

And

$$\rho = \frac{\operatorname{cov}(yz)}{\sqrt{v(y).v(z)}} \qquad -1 \le \rho \le 1$$

Taking square on b.s

$$\rho^2 = {[\operatorname{cov}(yz)]^2}/{v(y).v(z)} \qquad 0 \le \rho^2 \le 1$$

$$\rho^2 = \frac{(\tau'(\theta))^2}{\operatorname{var}(T(\underline{X})E\left[\frac{\partial \log L(\underline{X};\theta)}{\partial\theta}\right]^2}$$

AS $\rho^2 \le 1$

$$\frac{(\tau'(\theta))^2}{\operatorname{var}(T(\underline{X})E\left[\frac{\partial \log L(\underline{X};\theta)}{\partial\theta}\right]^2} \le 1$$

$$\frac{(\tau'(\theta))^2}{E\left[\frac{\partial \log L(\underline{X};\theta)}{\partial\theta}\right]^2} \le \operatorname{var}(T(\underline{X}))$$

OR

$$\operatorname{var}(T(\underline{X})) \ge \frac{(\tau'(\theta))^2}{E\left[\frac{\partial \log L(\underline{X};\theta)}{\partial\theta}\right]^2}$$

Which is known as Cramer Raw Inequality and Var(T) is called Cramer Raw Lower Bound(CRLB).

Question:

Define and write the importance and application of Cramer Rao Inequality

Solution:

Cramer Rao within the general class bound condition (such that the assumption concerning the regularity of the estimate ($\hat{\theta}$) are satisfy)then no estimator can have variance which is less then a quantity which depend upon f(x; θ)and n.If f($x_{1,}x_{2,}x_{3,}....,x_{n,};\theta)$ =t is an unbiased estimator of .

Then

$$\operatorname{var}(T) \ge {[\tau(\theta)]^2}/{n_i;(\theta)}$$

WHERE

$$n_i(\theta) = E\left[\frac{\partial \log L(\underline{X};\theta)}{\partial\theta}\right]^2 \qquad \text{(A)}$$

It is called the amount of information in the sample. Equation (A) is called the Cramer Rao Inequality and the right hand side is called Cramer Rao Lower Bound for variance of unbiased estimator of

$\tau(\theta)$.

This Inequality has two uses:

i) It gives a lower bound for the variance of unbiased estimators and experimenter using an unbiased estimator whose variance was close to the Cramer Rao Lower Bound would know that he was using a good unbiased estimator.

ii)If an unbiased estimator whose variance equal with the Cramer Rao Lower Bound can be found then this estimator is an uniformly Minimum Variance Unbiased Estimator.

Q.No. 4

Show that

a) $-E\left[\frac{\partial^2 \log L(\underline{X};\theta)}{\partial\theta^2}\right] = E\left[\frac{\partial \log L(\underline{X};\theta)}{\partial\theta}\right]^2$

Solution:

As we that the area under the curve is unity

$$\int L(\underline{X})dx = 1 \qquad \text{(A)}$$

Now partially differentiate w.r.t θ to equ (A)

$$\frac{\partial}{\partial\theta}\int L(\underline{X})dx = \frac{\partial}{\partial\theta}(1)$$

$$\int \frac{\partial}{\partial\theta}L(\underline{X})dx = 0$$

$$\int \frac{1}{L(\underline{X};\theta)}\frac{\partial}{\partial\theta}L(\underline{X};\theta)..L(\underline{X};\theta)dx = 0$$

$$\int \left\{\frac{\partial}{\partial\theta}\log L(\underline{X};\theta)\right\}L(\underline{X};\theta)d\theta = 0$$

Now again differentiate w.r.t ' θ'

$$\int \left\{\frac{\partial}{\partial\theta}\log L(\underline{X};\theta)\right\}\frac{\partial}{\partial\theta}L(\underline{X};\theta)dx + \int L(\underline{X};\theta)dx\frac{\partial}{\partial\theta}\left[\frac{\partial}{\partial\theta}\log L(\underline{X};\theta)\right] = 0$$

$$\int \frac{\partial}{\partial\theta}L(\underline{X};\theta)dx\frac{1}{L(\underline{X};\theta)}\frac{\partial}{\partial\theta}L(\underline{X};\theta)L(\underline{X};\theta)dx + \int L(\underline{X};\theta)dx\frac{\partial^2}{\partial\theta^2}\log L(\underline{X};\theta) = 0$$

$$\int \frac{\partial}{\partial\theta}\log L(\underline{X};\theta)\frac{\partial}{\partial\theta}\log L(\underline{X};\theta).L(\underline{X};\theta)dx + \int L(\underline{X};\theta)dx\left[\frac{\partial^2 \log L(\underline{X};\theta)}{\partial\theta^2}\right] = 0$$

$$\int \frac{\partial}{\partial\theta}\log L(\underline{X};\theta)\frac{\partial}{\partial\theta}\log L(\underline{X};\theta).L(\underline{X};\theta)dx + E\left[\frac{\partial^2 \log L(\underline{X};\theta)}{\partial\theta^2}\right] = 0$$

$$\int \left[\frac{\partial}{\partial\theta}\log L(\underline{X};\theta)\right]^2 .L(\underline{X};\theta)dx + E\left[\frac{\partial \log L(\underline{X};\theta)}{\partial\theta}\right] = 0$$

$$E\left[\frac{\partial \log L(\underline{X};\theta)}{\partial\theta}\right]^2 + E\left[\frac{\partial^2 \log L(\underline{X};\theta)}{\partial\theta^2}\right] == 0$$

$$-E\left[\frac{\partial^2 \log L(\underline{X};\theta)}{\partial\theta^2}\right] = E\left[\frac{\partial \log L(\underline{X};\theta)}{\partial\theta}\right]^2$$

Hence the required result.

(b) $$E[\frac{\partial \log L(\underline{X};\theta)}{\partial\theta}]^2 = nE[\frac{\partial \log F(X;\theta)}{\partial\theta}]^2$$

Let by definition

$$L(\underline{X};\theta) = \prod_{i=1}^{n} f(X;\theta)$$

Taking log on both sides

$$\log L(\underline{X};\theta) = \log[\prod_{i=1}^{n} f(X;\theta)]$$

$$\log L(\underline{X};\theta) = \sum \log f(X;\theta)$$

Differentiate w.r.t "θ"

$$\frac{\partial}{\partial\theta}\log L(\underline{X};\theta) = \frac{\partial}{\partial\theta}[\sum \log f(X;\theta)]$$

$$\frac{\partial}{\partial\theta}\log L(\underline{X};\theta) = \sum \frac{\partial}{\partial\theta}\log f(X;\theta)$$

Taking square on both sides

$$[\frac{\partial}{\partial\theta}\log L(\underline{X};\theta)]^2 = \frac{[\sum \partial \log f(X;\theta)]^2}{\partial\theta}$$

$$[\frac{\partial}{\partial\theta}\log L(\underline{X};\theta)]^2 = \sum^{n}(\frac{\partial}{\partial\theta}\log\ f(X;\theta))^2 + \sum_{i\neq j}^{n}\sum^{n-1}\frac{\partial}{\partial\theta}logf(X_i,\theta)\frac{\partial}{\partial\theta}\log\ f(X_j,\theta)$$

Applying expectation on both sides

$$E[\frac{\partial \log L(\underline{X};\theta)}{\partial\theta}]^2 = \sum^{n} E[\frac{\partial}{\partial\theta}\log\ f(X;\theta)]^2 + \sum_{i\neq j}^{n}\sum^{n-1} E[\frac{\partial}{\partial\theta}logf(X_i,\theta)\frac{\partial}{\partial\theta}\log\ f(X_j,\theta)]$$

AS X's are independent so their cross product term will be vanish

$$E[\frac{\partial \log L(\underline{X};\theta)}{\partial\theta}]^2 = nE[\frac{\partial}{\partial\theta}\log\ f(X;\theta)]^2$$

Hence the is required result.

Q.No. 5

Let $X_1, X_2, X_3....X_n$ be a random sample from a bernulli distribution whith parameter "θ". Then show that T=$\sum X$ is complete and sufficient for "θ". Hence produced UMVUE for "θ" and its variance attains Cramer Rao inequality lower Bound (CRLB).

Solution:

As X~ bernulli(θ)

$f(x) = (c_x^1)\theta^x(1-\theta)^{1-x}$ x:0,1

For completeness:

$X_1 + X_2 + ...X_n = \sum x = T \sim b(x, n, \theta)$

$f(t) = (c_t^n)\theta^t(1-\theta)^{n-t}$ t: 0,1,2,3,. . .,n

By definition of completeness

$E[h(t)] = 0$

$\sum h(t)f(t) = 0$

$\sum h(t)(c_t^n)\theta^t(1-\theta)^{n-t} = 0$

$then (c_t^n) \neq, \theta^t \neq 0, (1-\theta)^{n-t} \neq 0$

$only \sum h(t) = 0$

Hence it is proved T= $\sum X$ is complete for "θ" .

For sufficient:

$f(x) = (c_x^1)\theta^x(1-\theta)^{1-x}$

Taking likelihood function

$L(\underline{X};\theta) = f(\underline{X};\theta) = L(\underline{X}) = \prod_{i=1}^{n}[(c_x^1)\theta^x(1-\theta)^{1-x}]$

$= \theta^{\sum x}(1-\theta)^{n-\sum x}\prod_{i=1}^{n}(c_x^1)$

$= g(\sum X, \theta)h(\underline{X})$

Hence $h(\underline{X})$ is independent for parameter "θ" . therefore by Neymen Fisher Factorization T= $\sum X$ is sufficient for "θ".

For Unbiasedness:

As T= $\sum X$ ~ b(n;θ)

E(T)= $E(\sum X) == \sum E(X) == n\theta$

$E(T)/n = \theta$

$\theta = E(T)/n$

It means $\frac{T}{n} = \frac{\sum X}{n} = \overline{X}$ is an unbiased estimator of "θ" as $\sum X$ is sufficient and $\overline{X}$ is one to one function of $\sum X$ so $\overline{X}$ is also sufficient for "θ".

By Lehman Shaffe Theorem on estimator is uniform minimum variance unbiased estimator if it is unbiased, sufficient and complete. Hence $\overline{X}$ is UMVUE of "θ".

For cramer Raw Lower Bound:

$$\text{var}(t) \geq (\tau'(\theta))^2 \Big/ -E[\frac{\partial^2 \log L(\underline{X};\theta)}{\partial\theta^2}]$$

$$f(x) = (c_x^1)\theta^x(1-\theta)^{1-x}$$

Taking likelihood function

$$L(\underline{X};\theta) = f(\underline{X};\theta) = L(\underline{X}) = \prod_{i=1}^{n}[(c_x^1)\theta^x(1-\theta)^{1-x}]$$

$$= \prod_{i=1}^{n}(c_x^1)\theta^{\sum X}(1-\theta)^{n-\sum X}$$

Taking log likelihood function

$$\log L(\underline{X};\theta) = \log[\prod_{i=1}^{n}(c_x^1)\theta^{\sum X}(1-\theta)^{n-\sum X}]$$

$$= \sum X \log\theta + (n - \sum X)\log(1-\theta) + \sum \log(c_x^1)$$

Differentiate w.r.t "θ"

$$\frac{\partial}{\partial\theta}\log L(\underline{X};\theta) = \frac{\sum X}{\theta} + \frac{n-\sum X}{1-\theta}(-1) + 0$$

$$\frac{\partial}{\partial\theta}\log L(\underline{X};\theta) = \frac{\sum X}{\theta} - \frac{n-\sum X}{1-\theta} \qquad \text{(A)}$$

$$\frac{\partial}{\partial\theta}\log L(\underline{X};\theta) = \frac{(1-\theta)\sum X - \theta(n-\sum X)}{\theta(1-\theta)}$$

$$\frac{\partial}{\partial\theta}\log L(\underline{X};\theta) = \frac{1}{\theta(1-\theta)}[\sum X - \theta\sum X - n\theta + \theta\sum X]$$

$$\frac{\partial}{\partial\theta}\log L(\underline{X};\theta) = \frac{1}{\theta(1-\theta)}[n\overline{X} - n\theta]$$

$$\frac{\partial}{\partial\theta}\log L(\underline{X};\theta) = \frac{1}{\theta(1-\theta)}[\overline{X} - \theta] \qquad \text{(B)}$$

As we know that

$$\frac{\partial}{\partial\theta}\log L(\underline{X}) = A(\theta)\{\hat{\theta} - \tau(\theta)\}$$

$$A(\theta) = \frac{n}{\theta(1-\theta)} \qquad ,\hat{\theta} = \overline{X} \qquad ,\tau(\theta) = \theta \qquad \tau'(\theta) = 1$$

Again Differentiate w.r.t "θ" to eq (B)

$$\frac{\partial^2}{\partial\theta^2}\log L(\underline{X};\theta) = -\frac{\sum X}{\theta^2} + \frac{n-\sum X}{(1-\theta)^2}(-1)$$

$$\frac{\partial^2}{\partial\theta^2}\log L(\underline{X};\theta) = -\frac{\sum X}{\theta^2} - \frac{n+\sum X}{(1-\theta)^2}$$

Applying expectation on both sides

$$\frac{\partial^2}{\partial\theta^2}\log L(\underline{X};\theta) = E[-\frac{\sum X}{\theta^2} - \frac{n+\sum X}{(1-\theta)^2}]$$

$$\frac{\partial^2}{\partial\theta^2}\log L(\underline{X};\theta) = \frac{\sum E(X)}{\theta^2} - \frac{n-\sum E(X)}{(1-\theta)^2}$$

$$\frac{\partial^2}{\partial\theta^2}\log L(\underline{X};\theta) = \frac{-n\theta}{\theta^2} - \frac{n-n\theta}{(1-\theta)^2}$$

$$\frac{\partial^2}{\partial\theta^2}\log L(\underline{X};\theta) = \frac{-n}{\theta} - \frac{n(1-\theta)}{(1-\theta)^2}$$

$$\frac{\partial^2}{\partial\theta^2}\log L(\underline{X};\theta) = \frac{-n}{\theta} - \frac{n}{(1-\theta)}$$

$$\frac{\partial^2}{\partial\theta^2}\log L(\underline{X};\theta) = -n[\frac{1}{\theta} + \frac{1}{(1-\theta)}]$$

$$\frac{\partial^2}{\partial\theta^2}\log L(\underline{X};\theta) = -n[\frac{1-\theta+\theta}{\theta(1-\theta)}]$$

$$\frac{\partial^2}{\partial\theta^2}\log L(\underline{X};\theta) = \frac{-n}{\theta(1-\theta)}$$

As we know that For Cramer Raw Lower Bound (CRLB)

$$\operatorname{var}(t) \geq \frac{(\tau'(\theta))^2}{-E[\frac{\partial^2 \log L(\underline{X};\theta)}{\partial\theta^2}]}$$

Solve R.H.S

$$\frac{(\tau'(\theta))^2}{-E[\frac{\partial^2 \log L(\underline{X};\theta)}{\partial\theta^2}]} = \frac{1}{\frac{n}{\theta(1-\theta)}}$$

$$\frac{(\tau'(\theta))^2}{-E[\frac{\partial^2 \log L(\underline{X};\theta)}{\partial\theta^2}]} = \frac{\theta(1-\theta)}{n}$$

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As $V(T) \geq \frac{(\tau'(\theta))^2}{-E[\frac{\partial^2 \log L(\underline{X};\theta)}{\partial\theta^2}]}$

Hence it attains the Cramer Raw Lower Bound (CRLB)